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Estimates of Freud–Christoffel Functions for Some Weights with the Whole Real Line as Support

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Upper and lower bounds for generalized Christoffel functions, called Freud–Christoffel functions, are obtained. These have the form $\lambda_{n,p}(W, j, x) = \inf \|PW\|_{L_p(\mathbb{R})}/|P^{(j)}(x)|$, where the infimum is taken over all polynomials $P(x)$ of degree at most $n-1$. The upper and lower bounds for $\lambda_{n,p}(W, j, x)$ are obtained for all $0 < p \leqslant \infty$ and $j = 0, 1, 2, 3,\ldots$ for weights $W(x) = \exp(-Q(x))$, where, among other things, $Q(x)$ is bounded in $[-A, A]$, and Q'' is continuous in $\mathbb{R}\backslash(-A, A)$ for some $A > 0$. For $p = \infty$, the lower bounds give a simple proof of local and global Markov–Bernstein inequalities. For $p = 2$, the results remove some restrictions on Q in Freud's work. The weights considered include $W(x) = \exp(-|x|^\alpha/2)$, $\alpha > 0$, and $W(x) = \exp(-\exp(|x|^\rho))$, $\rho > 0$.

1. INTRODUCTION

Let $W(x)$ be a function, positive in $(-\infty, \infty)$, for which all moments $\int_{-\infty}^{\infty} W(x)\, x^j\, dx$, $j = 0, 1, 2,\ldots$, are finite. Let $p_n(W^2; x)$, $n = 0, 1, 2,\ldots$, be the sequence of orthonormal polynomials for $W^2(x)$; that is,

$$\int_{-\infty}^{\infty} p_n(W^2; x)\, p_m(W^2; x)\, W^2(x)\, dx = 1, \qquad m = n,$$

$$= 0, \qquad m \neq n,$$

$m, n = 0, 1, 2,\ldots$. The classical Christoffel functions are defined by

$$\lambda_n(W^2; x) = \inf_{P \in \mathbb{P}_{n-1}} \int_{-\infty}^{\infty} (P(u)\, W(u))^2\, du/(P(x))^2 \tag{1.1}$$

$$= \left\{\sum_{k=0}^{n-1} (p_k(W^2; x))^2\right\}^{-1}, \qquad n = 1, 2,\ldots, \tag{1.2}$$

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where $\mathbb{P}_{n-1}$ denotes the class of polynomials, with real coefficients, having degree at most $n-1$.

By approximating W and $1/W$ from below by polynomials, Freud [5–7] and Nevai [17, 18] estimated $\lambda_n(W^2; x)$ from above and below, with the aid of the Christoffel functions for the Legendre and Chebyshev weights on $[-1, 1]$. Typically, Freud [5–8] considered weights of the form $W(x) = \exp(-Q(x))$, where $Q(x)$ is convex and of polynomial growth as $|x| \to \infty$. Freud's upper bounds for $\lambda_n(W^2; x)$ [6] apply, for example, to $W_\alpha(x) = \exp(-|x|^\alpha/2)$, $\alpha > 1$, and his lower bounds [5, 7] apply to $W_\alpha(x)$, $\alpha \geqslant 2$. The case $\alpha = 1$ was investigated by Freud, Giroux, and Rahman [9].

The first lower bounds for $\lambda_n(W_\alpha^2; x)$ for $0 < \alpha < 1$ and $1 < \alpha < 2$, appear in Mhaskar and Saff [14, Theorem 6.5]. They credited their results for $1 < \alpha < 2$ to Freud, but gave a simplified and elegant proof. For $0 < \alpha < 1$, the first lower bounds appear in Mhaskar and Saff [14, Theorem 6.5]. It is interesting to note that since the moment problem for $W_\alpha^2(x)$ is indeterminate, $0 < \alpha < 1$,

$$1/\lambda_\infty(W_\alpha^2; x) = \sum_{k=0}^{\infty} (p_k(W_\alpha^2; x))^2$$

converges uniformly in compact subsets of $\mathbb{C}$ to an entire function of at most minimal type of order 1 (see Akhiezer [1, pp. 49–59]).

For special weights such as $W(x) = \exp(-x^4/2)$, precise asymptotic formulae were obtained for $\lambda_n(W^2; x)$ by Nevai [20, Theorem 2] and the bounds on the orthonormal polynomials in Bonan [3] and Nevai [21] trivially yield lower bounds for the Christoffel functions. The behaviour of Christoffel functions in $\mathbb{C}\backslash\mathbb{R}$ has been analyzed by Rahmanov [22].

Various generalizations of the classical Christoffel functions have been investigated by Freud [7] and Nevai [19]. In [7, pp. 23–24], Freud considered

$$\begin{aligned} \lambda_n(W^2; \Phi) &= \inf_{P \in \mathbb{P}_{n-1}} \int_{-\infty}^{\infty} (PW)^2(u)\, du/(\Phi(P))^2 \\ &= \left\{ \sum_{k=0}^{n-1} \{\Phi(p_k(W^2; x))\}^2 \right\}^{-1}, \end{aligned} \tag{1.3}$$

where Φ is an arbitrary linear functional defined on all polynomials. Further, he obtained lower bounds for $\lambda_n(W^2; \Phi)$ when $\Phi(P) = P^{(j)}(x)$, $j = 0, 1$, and together with some results on Cesaro means of partial sums of orthonormal expansions, used these to obtain a Markov–Bernstein inequality. In turn, the Markov–Bernstein inequality yielded lower bounds for $\lambda_n(W^2; \Phi)$ when $\Phi(P) = P^{(j)}(x)$, $j = 2, 3, 4, \ldots$. For Jacobi weights $u(x)$ on

$[-1, 1]$, Nevai [19, pp. 106–113] obtained upper and lower bounds throughout $[-1, 1]$ for the L_p Christoffel functions

$$\lambda_n(u, p, x) = \inf_{P \in \mathbb{P}_{n-1}} \int_{-1}^{1} |P(t)|^p u(t) \, dt / |P(x)|^p, \tag{1.4}$$

for all $0 < p < \infty$.

In this paper, we obtain upper and lower bounds for generalized Christoffel functions of the form

$$\lambda_{n,p}(W, j, x) = \inf_{P \in \mathbb{P}_{n-1}} \|PW\|_{L_p(\mathbb{R})} / |P^{(j)}(x)|, \tag{1.5}$$

for all $0 < p \leqslant \infty$ and $j = 0, 1, 2, \ldots$. We shall call these Freud–Christoffel functions. We use the formulation of the problem in (1.5) without the pth power of the norm, since it leads to unified proofs for all $0 < p \leqslant \infty$ for weights on $\mathbb{R}$. By contrast, Nevai showed (1.4) to be a more suitable formulation for weights on $[-1, 1]$. For $p = \infty$, the lower bounds for $\lambda_{n,\infty}(W, j, x)$ lead to a simple proof of local and global Markov–Bernstein inequalities, which does not require the lengthy process of [7].

The weights considered have the form $W(x) = \exp(-Q(x))$, where $Q(x)$ is bounded in each finite interval, and Q'' is continuous in $\mathbb{R} \backslash (-A, A)$, for some $A > 0$. Further Q is required to satisfy some additional conditions in some cases. The restrictions on Q are weaker than those in Freud [7]—for example, we do not require Q to be even, or convex, or of polynomial growth for large $|x|$.

Both the upper and lower bounds are new for all $p \neq 2$. For $p = 2$, the upper bounds are new if $j = 1, 2, 3, \ldots$ and the lower bounds are new for $j = 0$ for weights such as $W(x) = \exp(-\exp(|x|^\rho))$, $\rho > 0$, and are largely new for $W(x) = \exp(-|x|^\alpha/2)$, $1 < \alpha < 2$. The global Markov–Bernstein inequalities are also new for weights such as $W(x) = \exp(-\exp(|x|^\rho))$, while the local Markov–Bernstein inequalities are all new.

The proofs use ideas of Freud [5–7], Nevai [17–19], Mhaskar and Saff [14, 15] and a simple trick, which enables one to estimate $\lambda_{n,p}(W, j, x)$ from below in terms of $\lambda_{n,p}(W, j-1, x)$ for a large range of x. One interesting feature of the proofs is that we hardly use the theory of orthogonal polynomials, but deduce corollaries for orthogonal polynomials. Further, we exploit Cartan's lemma [2, p. 174], and a trick from the convergence theory of Padé approximation, as in Lubinsky [12]. In [12], inequalities relating L_p norms of weighted polynomials over finite and infinite intervals, were established. Related L_p inequalities have been obtained using different methods by Zalik [23, 24] and for a general class of weights by Mhaskar and Saff [14]. The classes of weights in [12, 14] overlap, but do not coincide.

For $p=\infty$ and $W_\alpha(x)$, $\alpha>0$, very precise and elegant inequalities of this type were obtained by Mhaskar and Saff [14] and they subsequently considered more general classes of weights [15, 16]. Using their methods and those of Rahmanov [22], one may obtain "asymptotically sharp" L_p inequalities for weights $W(x)=\exp(-Q(x))$, where $Q(x)$ is even, convex and continuous in $[0,\infty)$ (see [13]). Professor Saff has informed the author that he and Mhaskar have obtained sharp L_p inequalities in general situations.

The paper is set out as follows: In Section 2, we define our notation. In Section 3, the principal results are stated. In Section 4, the general lower bounds for $\lambda_{n,p}(W,j,x)$ are proved. In Section 5, some sums associated with the Chebyshev polynomials are estimated. In Section 6 the main upper bounds are proved. Finally, in Section 7, the weights $\exp(-|x|^\alpha)$, $\alpha>0$, and $\exp(-\exp(|x|^\rho))$, $\rho>0$, are considered.

The author would like to thank the referee for his patient and meticulous reading of the original version of the manuscript, and for his suggestions to improve the standard of presentation. In the original version of this paper, integral inequalities were established, and these are referred to in [12]. These inequalities are omitted from the present version, but this does not affect the proofs in [12].

2. Notation

Throughout, $W(x)$ denotes a function positive in $(-\infty,\infty)$ and $Q(x)=\log 1/W(x)$, $x\in(-\infty,\infty)$. We usually assume

$$Q(x) \text{ is bounded in each finite interval, and } \lim_{|x|\to\infty} Q(x)/\log|x|=\infty, \tag{2.1}$$

so that all moments of $W(x)$ are finite. Further, we assume that

$$Q''(x) \text{ is continuous in } (-\infty,\infty)\backslash(-A,A) \text{ for some } A>0 \text{ and } Q'' \text{ is not identically zero there.} \tag{2.2}$$

Throughout, for $j=0, 1, 2$, we let

$$M_j(\xi)=\max\{|Q^{(j)}(u)|: A\leqslant|u|\leqslant\xi\}, \qquad \xi>A. \tag{2.3}$$

The orthonormal polynomials associated with the weight W^2 are denoted by $p_n(W^2;x)$, $n=0, 1, 2,\ldots$, and satisfy

$$\begin{aligned}\int_{-\infty}^{\infty} p_n(W^2;x)\,p_m(W^2;x)\,W^2(x)\,dx &= 1, \qquad m=n,\\ &=0, \qquad m\neq n,\end{aligned}$$

$m, n = 0, 1, 2, \ldots$ The class of polynomials of degree at most n, with real coefficients, is denoted by $\mathbb{P}_n$. Given $0 < p \leqslant \infty$, $-\infty \leqslant a < b \leqslant \infty$, and a measurable function $g(x)$ on (a, b), we let

$$\|g\|_{L_p(a,b)} = \left(\int_a^b |g(x)|^p\,dx\right)^{1/p}, \qquad 0 < p < \infty,$$

$$= \operatorname{ess\,sup}\{|g(x)| : x \in (a, b)\}, \qquad p = \infty.$$

Given $0 < p \leqslant \infty$, $j = 0, 1, 2, \ldots$, and $n = j+1, j+2, \ldots$, the Freud–Christoffel functions are

$$\lambda_{n,p}(W, j, x) = \inf_{P \in \mathbb{P}_{n-1}} \|PW\|_{L_p(-\infty,\infty)}/|P^{(j)}(x)|, \qquad x \in (-\infty, \infty). \tag{2.4}$$

For the special case $p = 2$, (1.3) yields

$$\lambda_{n,2}(W, j, x) = \left\{\sum_{k=0}^{n-1} (p_k^{(j)}(W^2; x))^2\right\}^{-1/2}, \tag{2.5}$$

$j = 0, 1, 2, \ldots; n = j+1, j+2, \ldots$

The iterated exponential function $\exp_j(x)$ and the iterated logarithm $\log_j(x)$ are defined as follows:

$$\exp_0(x) = x, \qquad x \in (-\infty, \infty),$$

$$\exp_j(x) = \exp(\exp_{j-1}(x)), \qquad x \in (-\infty, \infty), j = 1, 2, 3, \ldots,$$

while

$$\log_0(x) = x, \qquad x \in (-\infty, \infty),$$

$$\log_j(x) = \log(\log_{j-1}(x)), \qquad x \in (\exp_{j-1}(0), \infty), j = 1, 2, 3, \ldots.$$

Throughout, $C, C_1, C_2, \ldots$ denote positive constants independent of n and x, which are not necessarily the same from line to line. When stating inequalities for polynomials P of degree at most n, $C, C_1, C_2, \ldots$ will denote constants independent of P and n. To denote dependence of constants C on parameters $\varepsilon, j, \ldots$, we shall write $C = C(\varepsilon, j)$ and so on.

The usual symbols $\sim$, o, O will be used to compare functions or sequences. Thus, $f(x) \sim g(x)$ if for some C_1 and C_2, $C_1 \leqslant f(x)/g(x) \leqslant C_2$ for all x considered. Further, given two sequences $\{a_n\}$ and $\{b_n\}$, we say $a_n \sim b_n$ if for some C_1 and C_2, and all large enough n, $C_1 \leqslant a_n/b_n \leqslant C_2$. The symbol D_x^j will denote the jth derivative with respect to the independent variable, when that independent variable is taken to be x. Thus, for example, $D_x^j f(t) = f^{(j)}(x)$.

3. Statement of Results

Our main theorem on lower bounds for $\lambda_{n,p}(W, j, x)$ is the following:

THEOREM 3.1. *Let* $W(x) = \exp(-Q(x))$, *where* Q *satisfies* (2.1) *and* (2.2). *For large enough positive* x, *let* ξ_x *denote the positive root of the equation*

$$\xi_x^2 M_2(\xi_x) = x. \tag{3.1}$$

Let $0 < p \leqslant \infty$. *Let* s *be a positive integer and* j *be a non-negative integer.*

(i) *Given* $0 < \varepsilon < 1$,

$$\lambda_{n,p}(W, j, x) \geqslant C(\xi_n/n)^{j+1/p} W(x), \tag{3.2}$$

for all $|x| \leqslant \varepsilon \xi_{sn}$, *where* $C = C(\varepsilon, j, p, s)$.

(ii) *For all large enough* n, *let*

$$\mu_n = (1 - \xi_{sn}/\xi_{2sn})^{1/2}. \tag{3.3}$$

Then

$$\lambda_{n,p}(W, j, x) \geqslant C(\xi_n \mu_n/n)^{j+1/p} W(x), \tag{3.4}$$

for all $|x| \leqslant \xi_{sn}$, *where* $C = C(j, p, s)$.

Remarks. (i) The only condition on Q above is that $Q''(x)$ be continuous for large $|x|$, and Q be bounded in each finite interval. Thus the above result weakens Freud's restrictions [6, 7] that $Q(x)$ be convex and of polynomial growth.

(ii) When Q'' is of smooth polynomial growth, one can show $\mu_n \sim 1$. Further, one can replace ξ_{2sn} in (3.3) by ξ_{cn}, where c is arbitrarily large. This might ensure $\mu_n \sim 1$ in slightly more general situations.

COROLLARY 3.2. *Assume the conditions of Theorem* 3.1.

(i) *Local Markov–Bernstein inequality*: *Let* $0 < \delta < \varepsilon \leqslant 1$. *Then*

$$\|P'W\|_{L_\infty(-\delta\xi_{sn}, \delta\xi_{sn})} \leqslant C(n/\xi_n) \|PW\|_{L_\infty(-\varepsilon\xi_{sn}, \varepsilon\xi_{sn})}, \tag{3.5}$$

for all polynomials P *of degree* $\leqslant n$.

(ii) *Let* $0 < \delta < 1$ *and* j *be a non-negative integer. Then for* $|x| \leqslant \delta\xi_{sn}$,

$$W^2(x) \sum_{k=0}^{n-1} (p_k^{(j)}(W^2; x))^2 \leqslant C(n/\xi_n)^{2j+1}. \tag{3.6}$$

Although (i) of Corollary 3.2 does not follow immediately from Theorem 3.1, its proof is contained in that of Theorem 3.1. One can also prove a global Markov–Bernstein inequality:

COROLLARY 3.3. *Assume the conditions of Theorem* 3.1. *Further assume we are given a sequence* $\{\kappa(n)\}_{n=1}^{\infty}$ *of positive integers such that* $\kappa(n)\geqslant n$, *and such that for all polynomials* $P(x)$ *of degree* $\leqslant n$,

$$\|PW\|_{L_\infty(\mathbb{R})}\leqslant 2\,\|PW\|_{L_\infty(-\xi_{\kappa(n)},\xi_{\kappa(n)})}. \tag{3.7}$$

Then

(i) *For* $0<p\leqslant\infty$ *and* $j=0,1,2,...,$

$$\lambda_{n,p}(W,j,x)\geqslant C(\xi_n\mu_{\kappa(n)}/\kappa(n))^{j+1/p}W(x),\qquad x\in\mathbb{R}. \tag{3.8}$$

(ii) *Global Markov–Bernstein Inequality*:

For all polynomials P *of degree* $\leqslant n$,

$$\|P'W\|_{L_\infty(\mathbb{R})}\leqslant C(\kappa(n)/(\xi_n\mu_{\kappa(n)}))\,\|PW\|_{L_\infty(\mathbb{R})}. \tag{3.9}$$

When Q is of smooth polynomial growth, one may replace $\xi_n\mu_{\kappa(n)}/\kappa(n)$ by ξ_n/n in (3.8) and (3.9). More precisely, this is the case whenever Q has the following properties:

I. Whenever $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy $\alpha_n\to\infty$ and $\alpha_n/\beta_n\to 1$, as $n\to\infty$, then

$$M_2(\alpha_n)/M_2(\beta_n)\to 1\qquad\text{as } n\to\infty.$$

II. There exist $C_1>0$ and $C_2>0$ such that for $\xi\geqslant C_1$,

$$3\xi^2M_2(\xi)(\log(|x|/\xi))Q(x)\leqslant 1,\qquad |x|\geqslant C_2\xi.$$

When Q is of faster than polynomial growth, it is not obvious whether Corollary 3.3 can be improved.

Our main upper bound is the following:

THEOREM 3.4. *Let* $W(x)=\exp(-Q(x))$, *where* $Q(x)$ *satisfies* (2.1) *and* (2.2), *and satisfies the following additional assumptions*:

I. *For each* $K>0$,

$$|\min\{0,\min\{Q''(u): A\leqslant|u|\leqslant K\xi\}\}|=o(M_1(\xi)/\xi),\quad \xi\to\infty. \tag{3.10}$$

II. *For each* $\eta>0$, *there exist* $\varepsilon>0$ *and* $C>0$ *such that*

$$M_1(\varepsilon\xi)/M_1(\xi)<\eta,\qquad \xi>C. \tag{3.11}$$

III. *There exist* $C_1>0$ *and* $C_2>0$ *such that for* $\xi \geqslant C_1$,

$$3\xi M_1(\xi)\{\log(|x|/\xi)\}/Q(x) \leqslant 1, \qquad |x| \geqslant C_2\xi. \tag{3.12}$$

For large enough positive x, *let* q_x *denote the root of the equation*

$$q_x M_1(q_x) = x. \tag{3.13}$$

Then for $0<p\leqslant\infty$, *and* $j=0, 1, 2,...,$

$$\lambda_{n,p}(W, j, x) \leqslant C_3(q_n/n)^{j+1/p} W(x), \tag{3.14}$$

$|x| \leqslant C_4 q_n$, *where* $C_3, C_4 = C_3, C_4(j, p)$.

Remarks. (a) The implicit conditions (3.10), (3.11), and (3.12) are rather unattractive, but seem essential in some form in the proofs. Note that (3.10) holds if $Q(x)$ is convex for $|x| \geqslant A$, since then the left member of (3.10) is 0. Further, both (3.11) and (3.12) seem to be satisfied if $Q'(x)$ grows faster than some power of x, and does not behave too wildly. For the special, but important, case $j=0$, one does not need (3.11). Further (3.10) and (3.12) may be weakened but we omit the rather awkward general formulation.

(b) The definition (3.13) of q_x is a natural generalization of Freud's q_x [5–7].

(c) The restrictions on Q above weaken those of [6] where Q was required to be convex, and had to satisfy

$$1+C_1 < Q'(2x)/Q'(x) < 1+C_2, \qquad x > C_3,$$

which forces Q to be of polynomial growth.

(d) The above results may be stated in a more explicit form for weights such as $W(x)=\exp(-\exp(|x|^\rho))$ or $W(x)=\exp(-|x|^\alpha/2)$:

THEOREM 3.5. *Let* l *be a positive integer, let* $c, \rho>0$, *and*

$$W(x)=\exp(-c\exp_l(|x|^\rho)), \qquad x\in\mathbb{R}. \tag{3.15}$$

For large enough n, *let*

$$\theta_n = (\log_l n)^{1/\rho}. \tag{3.16}$$

(i) *Then for* $0<p\leqslant\infty$ *and* $j=0, 1, 2,...,$

$$\lambda_{n,p}(W, j, x) \sim (\theta_n/n)^{j+1/p} W(x), \tag{3.17}$$

for $|x| \leqslant C\theta_n$, *where* $C=C(j,p)$.

The lower bound in $\sim$ *holds for all* $|x| \leqslant \delta\theta_n$, *any* $0<\delta<1$.

(ii) *For large enough n, let*

$$v_n = \left(\prod_{k=1}^{l} \log_k n \right)^2. \tag{3.18}$$

Then for $0<p\leqslant\infty$ *and* $j=0, 1, 2, \ldots,$

$$\lambda_{n,p}(W, j, x) \geqslant C(\theta_n/(nv_n))^{j+1/p} W(x), \qquad x\in\mathbb{R}, \tag{3.19}$$

where $C=C(j,p)$.

(iii) *For any polynomial* $P(x)$ *of degree* $\leqslant n$,

$$\|P'W\|_{L_\infty(\mathbb{R})} \leqslant C(nv_n/\theta_n)\, \|PW\|_{L_\infty(\mathbb{R})}. \tag{3.20}$$

(iv) *For all non-negative integers j,*

$$W^2(x) \sum_{k=0}^{n-1} \{p_k^{(j)}(W^2; x)\}^2 \leqslant C(nv_n/\theta_n)^{2j+1}, \qquad x\in\mathbb{R}. \tag{3.21}$$

It seems probable that the exponent 2 in (3.18) can be replaced by $\frac{1}{2}$. In any event, the above results are the first to appear for these weights. Noting that $\lim_{n\to\infty} \xi_n/\theta_n = 1$, Corollary 3.2(i) yields better results for intervals of the form $(-\delta\theta_n, \delta\theta_n)$, $0<\delta<1$.

THEOREM 3.6. *Let* $W(x)=\exp(-|x|^\alpha/2)$, $x\in\mathbb{R}$, $\alpha>0$. *Let*

$$\theta_n = n^{1/\alpha}, \qquad n=1, 2, \ldots. \tag{3.22}$$

(i) *If* $\alpha\geqslant 2$, *then for* $0<p\leqslant\infty$ *and* $j=0, 1, 2, \ldots,$

$$\lambda_{n,p}(W, j, x) \sim (\theta_n/n)^{j+1/p} W(x), \tag{3.23}$$

for $|x|\leqslant C\theta_n$, *where* $C=C(j,p)$. *Further,*

$$\lambda_{n,p}(W, j, x) \geqslant C(\theta_n/n)^{j+1/p} W(x), \qquad x\in\mathbb{R}, \tag{3.24}$$

where $C=C(j,p)$.

(ii) *If* $1<\alpha<2$, *then for* $0<p\leqslant\infty$ *and* $j=0, 1, 2, \ldots,$

$$\lambda_{n,p}(W, j, x) \leqslant C(\theta_n/n)^{j+1/p} W(x), \tag{3.25}$$

for $|x|\leqslant C_1\theta_n$, *where* $C, C_1 = C, C_1(j,p)$. *Further, given* $\varepsilon>0$,

$$\lambda_{n,p}(W, j, x) \geqslant C(\theta_n/n)^{j+1/p} W(x), \tag{3.26}$$

for $|x|\geqslant \varepsilon\theta_n$, *where* $C=C(\varepsilon, j, p)$.

(iii) *If* $0<\alpha<1$, *then for* $j=0$ *and* $0<p\leqslant\infty$, *and for* $j=0, 1, 2,...$ *and* $0<p\leqslant 2$, *we have*

$$\lambda_{n,p}(W, j, x)\geqslant CW(x),\ x\in\mathbb{R}, \tag{3.27}$$

where $C=C(j, p)$.

Remarks. (a) The lower bounds in (i) are contained in Freud [7] for the case $p=2$ and $j=0, 1, 2,....$ The upper bounds in (i) are contained in Freud [6] for the case $p=2$ and $j=0$, but are otherwise new.

(b) The upper bounds in (ii) are contained in Freud [6] for the case $p=2$ and $j=0$, but are otherwise new. The lower bounds in (ii) are new, but Mhaskar and Saff [14, Theorem 6.5] established (and credited Freud) with the estimate

$$\lambda_{n,2}(W, 0, x)\geqslant C(\theta_n/(n\log n))^{1/2}W(x), \qquad x\in\mathbb{R}.$$

We note that the proof of (3.26) actually establishes (3.26) for all $0<\alpha<2$, but (3.26) is of interest mainly for $\alpha>1$. For $\alpha=1$, Freud, Giroux and Rahman [9] showed

$$\lambda_{n,2}(W, 0, x)\geqslant C(\log n)^{-1/2}W(x), \qquad x\in\mathbb{R}.$$

(c) The lower bounds in (iii) are new. It seems likely that (3.27) holds for all $0<p\leqslant\infty$, and $j=0, 1, 2,...$, but the author could not prove it. In [14, Theorem 6.5], Mhaskar and Saff stated $\lambda_{n,2}(W, 0, x)\geqslant CW(x)/n^{1/2}$, $x\in\mathbb{R}$, but their proof actually shows that for all $x\in\mathbb{R}$,

$$W^{-1}(x)\,\lambda_{n,2}(W, 0, x)\geqslant\lambda_{n,2}(W, 0, 0)\geqslant C.$$

(d) An interesting consequence of (3.27) for $j=0$ and $0<p<\infty$ is that the polynomials are not dense in the weighted space $\Lambda_p=\{f: fW\in L_p(\mathbb{R})\}$. Although this seems to be known if $p=2$, the author could not ascertain whether it is known for $p\neq 2$. The details are contained in Lemma 7.5.

(e) Theorems 3.1 and 3.4 can be applied to weights such as $W(x)=(1+x^2)^\beta\exp(-|x|^\alpha/2)$, which have been considered by Zalik [23, 24] for $\alpha=2$. Further, the proofs of Theorem 3.1 and 3.4 can be modified to handle weights such as $W(x)=|x|^\beta\exp(-|x|^\alpha/2)$, by using the upper and lower bounds in Nevai's memoir [19] for the Christoffel functions for the weight $w(x)=|x|^\beta$, $x\in[-1, 1]$.

4. Proof of the General Lower Bound

The idea of Freud [5, 6] and Nevai [17, 18] to find lower bounds for $\lambda_{n,2}(W, 0, x)$ is to approximate $W(x)$ from below by polynomials and then to apply estimates for the Christoffel functions for the Legendre weight on $[-1, 1]$. We proceed with the construction of suitable polynomials.

Lemma 4.1. *Let m be a positive integer, and*

$$P_m(x) = \sum_{j=0}^{m} x^j/j!. \tag{4.1}$$

Then

$$(3/4)\exp(x) \leqslant P_m(x) \leqslant (5/4)\exp(x), \qquad |x| \leqslant m/4. \tag{4.2}$$

Proof. Let $\Delta(x) = \exp(x) - P_m(x)$. This function has a zero of multiplicity $m+1$ at $x=0$, and so $\Delta(x)/x^{m+1}$ is entire. Applying Cauchy's integral formula to this latter function, we obtain for $|x| \leqslant m/4$,

$$\begin{aligned}|\Delta(x)| &= \left|\frac{1}{2\Pi i}\int_{|t| = m\log 4} \frac{\exp(t)}{t-x}\left(\frac{x}{t}\right)^{m+1} dt\right| \\ &\leqslant \left(\frac{m\log 4}{m\log 4 - m/4}\right)\exp(m\log 4)(4\log 4)^{-(m+1)} \\ &\leqslant \exp(-m/4)/4 \leqslant \exp(-x)/4.\end{aligned}$$

Now (4.2) follows. ∎

Before completing the construction of the polynomials, we need the basic properties of ξ_x:

Lemma 4.2. *Let $W(x) = \exp(-Q(x))$, where Q satisfies* (2.1) *and* (2.2). *Let ξ_x be defined by* (3.1) *for large positive x. Then, for large positive x,*

(i) *ξ_x is non-decreasing, and continuous.*

(ii) $\lim_{x\to\infty} \xi_x = \infty$.

(iii) $\xi_x \leqslant Cx^{1/2}$.

(iv) $\xi_{2x}/\xi_x \leqslant 2^{1/2}$.

Proof. Part (i) follows from (3.1) and the fact that $\xi^2 M_2(\xi)$ is continuous and strictly increasing in ξ;

(ii) follows from the fact that $\xi^2 M_2(\xi)$ is bounded in finite intervals;

(iii) follows from (3.1) and (2.2) which show that $M_2(\xi)$ is not identically zero;

(iv) follows from (3.1): By (3.1),

$$(\xi_{2x}/\xi_x)^2 = 2M_2(\xi_x)/M_2(\xi_{2x}) \leqslant 2. \quad \blacksquare$$

We can now, following Freud [7], complete the construction of the polynomials.

LEMMA 4.3. *Let* $W(x) = \exp(-Q(x))$, *where* Q *satisfies* (2.1) *and* (2.2). *Further, let us assume*

$$Q'(0) = 0, \tag{4.3}$$

and that $A = 0$ *in* (2.2), *so that*

$$Q'' \text{ is continuous in } \mathbb{R}. \tag{4.4}$$

For each fixed x, *define a quadratic in* t *by*

$$S_n[t; x] = Q'(x)(t-x) + M_2(\xi_n)(t-x)^2/2, \tag{4.5}$$

and with the notation of (4.1), *let*

$$R_n[t; x] = W(x)\, P_{16n}(-S_n[t; x]). \tag{4.6}$$

Then

(i) $R_n[x; x] = W(x)$ *and* $D_x R_n[t; x] = -Q'(x)\, W(x)$. (4.7)

(ii) $R_n[t; x]$ *has degree* $32n$.

(iii) $W(t) \geqslant (4/5)\, R_n[t; x] > 0, \quad |x| \leqslant \xi_n,\ |t| \leqslant \xi_n.$ (4.8)

Proof. Parts (i) and (ii) follow immediately from (4.1), (4.5), and (4.6).

(iii) By (4.3), we have for $|x| \leqslant \xi_n$,

$$|Q'(x)| = \left| \int_0^x Q''(u)\, du \right| \leqslant \xi_n M_2(\xi_n). \tag{4.9}$$

Then (4.5) and (4.9) yield for $|x|, |t| \leqslant \xi_n$,

$$|S_n[t; x]| \leqslant 2\xi_n^2\, M_2(\xi_n) + 2M_2(\xi_n)\, \xi_n^2 = 4n.$$

Then by Lemma 4.1, for $|x|, |t| \leqslant \xi_n$,

$$\begin{aligned} 0 &< W(x)\, P_{16n}(-S_n[t; x]) \\ &\leqslant (5/4) \exp(-Q(x) - S_n[t; x]) \\ &= (5/4)\, W(t) \exp(Q(t) - Q(x) - Q'(x)(t-x) - M_2(\xi_n)(t-x)^2/2) \end{aligned}$$

(by (4.5))

$$= (5/4)\, W(t) \exp((Q''(\theta) - M_2(\xi_n))(t-x)^2/2)$$

(where θ lies between t and x)

$$\leqslant (5/4)\, W(t).$$

Finally (4.6) and this last inequality yield (4.8). ∎

The next few lemmas will be used to estimate $\lambda_{n,p}(W, j, x)$ from below by induction on j.

LEMMA 4.4. *For all polynomials $T(u)$ of degree $\leqslant n$, and for $\xi > 0$, $|x| < \xi$,*

$$|T'(x)| \leqslant (n/\xi)(1-(x/\xi)^2)^{-1/2}\, \|T\|_{L_\infty[-\xi,\xi]}. \tag{4.10}$$

Proof. Apply Theorem 3 in Lorentz [11, p. 39] to $P(u) = T(u\xi)$. ∎

Finally, we need the following lemma before proving Theorem 3.1:

LEMMA 4.5. *Let $0 < p \leqslant \infty$. Then for all $\xi > 0$, $|x| < \xi$, and all polynomials P of degree $\leqslant n-1$,*

$$\|P\|_{L_p(-\xi,\xi)}/|P(x)| \geqslant C(1-(x/\xi)^2)^{1/(2p)}(\xi/n)^{1/p}, \tag{4.11}$$

where $C = C(p)$ only.

Proof. If $p = \infty$, the left member of (4.11) is bounded below by 1, while the right is just C, so if $p = \infty$, we may take $C = 1$. Suppose now $0 < p < \infty$. It follows from results in Nevai's memoir (see Definition 6.3.1 [19, p. 106], Definition 6.3.4 [19, p. 107], Lemma 6.3.5 [19, p. 108] and Theorem 6.3.13 [19, p. 113]) that for all polynomials $T(u)$ of degree $\leqslant n-1$, $T \not\equiv 0$, and for all $|y| < 1$,

$$\int_{-1}^{1} |T(u)|^p\, du/|T(y)|^p \geqslant \lambda_n(dx, p, y)$$

(with Nevai's notation)

$$\geqslant C((1-y)^{1/2} + 1/n)((1+y)^{1/2} + 1/n)/n$$

$$\geqslant C(1-y^2)^{1/2}/n.$$

Here $C = C(p)$. If we apply this inequality to the polynomial $T(u) = P(\xi u)$, where P is a polynomial of degree $\leqslant n-1$, make a substitution in the integral defining the p-norm and let $y = x/\xi$, we obtain (4.11). ∎

Proof of Theorem 3.1. We note first that we may assume that both (4.3) and (4.4) hold. For else, we can define $Q^*(u)$ satisfying (4.3) and (4.4) (with Q^* replacing Q there), and such that $Q^*(u) = Q(u)$ for $|u| \geqslant A$. Then as $Q(u)$ and $Q^*(u)$ are bounded in $|u| \leqslant A$,

$$W^*(x) \triangleq \exp(-Q^*(x)) \sim \exp(-Q(x)) = W(x),$$

for $x \in \mathbb{R}$. Consequently for all $x \in \mathbb{R}$,

$$\lambda_{n,p}(W^*, j, x) \sim \lambda_{n,p}(W, j, x). \tag{4.12}$$

Further, changing $Q(u)$ for $|u| \leqslant A$ does not affect the asymptotic behaviour of ξ_x: To distinguish the quantities for Q and Q^*, let

$$M_2^*(\xi) = \max\{|(Q^*)''(u)|: |u| \leqslant \xi\},$$

and let ξ_x^* be defined by

$$(\xi_x^*)^2 M_2^*(\xi_x^*) = x, \tag{4.13}$$

for large positive x.

If, firstly, $M_2(\xi) \to \infty$ as $\xi \to \infty$, then we see $M_2^*(\xi) = M_2(\xi)$ for large ξ and consequently $\xi_x^* = \xi_x$, for large x. In this case, it obviously suffices to prove Theorem 3.1 for Q^* and W^*.

Suppose next, $M_2(\xi)$ remains bounded (and hence approaches a finite limit) as $\xi \to \infty$. Then the same is true of $M_2^*(\xi)$. We deduce from (3.1) and (4.13) that $\xi_x^* \sim \xi_x \sim x^{1/2}$, for large x. Further, (3.1) shows that

$$(\xi_{2sn}/\xi_{sn})^2 = 2M_2(\xi_{sn})/M_2(\xi_{2sn}) \to 2,$$

as $n \to \infty$. Hence $\mu_n \sim 1$ in (3.3) and, similarly, $\mu_n^* \sim 1$, where μ_n^* is the analogous quantity for Q^*. If we can prove Theorem 3.1 for W^* and Q^*, then given a positive integer s^*, (3.4) and (4.12) imply that

$$\lambda_{n,p}(W, j, x) \geqslant C(\xi_n/n)^{j+1/p} W(x),$$

for $|x| \leqslant \xi_{s^*n}^*$, and hence for $|x| \leqslant C(s^*n)^{1/2}$. If we take s^* large enough, we shall have proved both (3.2) and (3.4) for $|x| \leqslant \xi_{sn}$.

Thus, we can assume (4.3) and (4.4), and now proceed with the proof. For $0 \leqslant a < 1$, let

$$r(a) = (1-a^2)^{1/2}. \tag{4.14}$$

We first prove the following statement by induction on j: For all $0<a<1$, and for all $|x| \leqslant a\xi_{sn}$,

$$W^{-1}(x)\,\lambda_{n,p}(W,j,x) \geqslant C(ar(a)\,\xi_{sn}/n)^{j+1/p}, \tag{4.15}$$

where $C=C(j,p,s)$ but $C \neq C(a,n,x)$.

$j=0$: Let $R_{sn}[t;x]$ be the polynomial of degree $32sn$, defined by (4.5) and (4.6) with sn replacing n. Then for $|x|<\xi_{sn}$, (2.4), (4.7) and (4.8) yield

$$\begin{aligned}
&W^{-1}(x)\,\lambda_{n,p}(W,o,x)\\
&\quad\geqslant (4/5)\inf_{\mathbb{P}_{n-1}} \|P(t)\,R_{sn}[t;x]\|_{L_p(-\xi_{sn},\xi_{sn})}/|P(x)\,R_{sn}[x;x]|\\
&\quad\geqslant C(1-(x/\xi_{sn})^2)^{1/(2p)}(\xi_{sn}/(32sn+n))^{1/p}\\
&\qquad\text{(by Lemma 4.5, where } C=C(p)\text{ only)}\\
&\quad\geqslant C(r(a)\,\xi_{sn}/n)^{1/p},\\
&\qquad\text{for } |x|\leqslant a\xi_{sn},\text{ with } C=C(s,p).
\end{aligned}$$

Assume true for $0, 1, 2,..., j-1 (j \geqslant 1)$: Let $0<a<1$ and let $b=a^{1/2}$. Note that, by (4.14),

$$br(b)=b(1-a)^{1/2}=b(1-a^2)^{1/2}/(1+a)^{1/2}>ar(a)/2. \tag{4.16}$$

Now by (2.4), for $|x| \leqslant a\xi_{sn}$,

$$\begin{aligned}
&W^{-1}(x)\,\lambda_{n,p}(W,j,x)\\
&\quad\geqslant \inf_{\mathbb{P}_{n-1}}\left\{\frac{\|PW\|_{L_p(\mathbb{R})}}{\|P^{(j-1)}W\|_{L_\infty(-b\xi_{sn},b\xi_{sn})}}\,\frac{\|P^{(j-1)}W\|_{L_\infty(-b\xi_{sn},b\xi_{sn})}}{|P^{(j)}(x)\,W(x)|}\right\}\\
&\quad\geqslant C(br(b)\,\xi_{sn}/n)^{j-1+1/p}\inf_{\mathbb{P}_{n-j}}\|PW\|_{L_\infty(-b\xi_{sn},b\xi_{sn})}/|P'(x)\,W(x)|,
\end{aligned} \tag{4.17}$$

by induction hypothesis, that is, by (4.15) with j replaced by $j-1$. Next, let $P \in \mathbb{P}_{n-j}$ and let $|x| \leqslant a\xi_{sn}=b^2\xi_{sn}$. By (4.7),

$$\begin{aligned}
|P'(x)\,W(x)| &= |P'(x)\,R_{sn}[x;x]|\\
&= |D_x(P(t)\,R_{sn}[t;x])-P(x)\,D_xR_{sn}[t;x]|
\end{aligned}$$

(here D_x denotes the derivative w.r.t. t at $t=x$)

$$\leqslant C(n/(br(b)\,\xi_{sn}))\,\|PR_{sn}\|_{L_\infty(-b\xi_{sn},b\xi_{sn})}+|Q'(x)|\,|P(x)\,R_{sn}[x;x]|$$

(by Lemma 4.4 and by (4.7))

$$\leqslant C\,\|PW\|_{L_\infty(-b\xi_{sn},b\xi_{sn})}\{(n/(br(b)\,\xi_{sn}))+(sn/\xi_{sn})\}, \tag{4.18}$$

by (4.8), and as at (4.9). Substituting this last inequality into (4.17) and using (4.16), we obtain (4.15) for j. This completes the inductive proof of (4.15).

Finally, Theorem 3.1(i) follows from (4.15) by taking $a=\varepsilon$, while Theorem 3.1(ii) follows from (4.15) if we replace s by $2s$ so that (4.15) holds for $|x|\leqslant a\xi_{2sn}$. If we let $a=\xi_{sn}/\xi_{2sn}$, then $r(a)\geqslant\mu_n$, where the latter is defined by (3.3), while $a\xi_{2sn}=\xi_{sn}$. ▮

Proof of Corollary 3.2(i). The statement (3.5) is equivalent to the following:

$$\inf_{\mathbb{P}_n}\|PW\|_{L_\infty(-\varepsilon\xi_{sn},\varepsilon\xi_{sn})}/|P'W(x)|\geqslant C\xi_n/n,$$

$|x|\leqslant\delta\xi_{sn}$. This may be proved in exactly the same way as the induction step in the proof of (4.15); see (4.18). ▮

Proof of Corollary 3.2(ii). This follows from (2.5) and Theorem 3.1(i). ▮

Proof of Corollary 3.3. Since $\lambda_{n,p}$ is monotone decreasing in n, and since $\kappa(n)\geqslant n$, (2.4) and Theorem 3.1(ii) show that for $|x|\leqslant\xi_{\kappa(n)}$ and $P\in\mathbb{P}_{n-1}$,

$$\begin{aligned}|P^{(j)}W|(x)/\|PW\|_{L_p(\mathbb{R})}&\leqslant\lambda_{n,p}^{-1}(W,j,x)\,W(x)\\&\leqslant\lambda_{\kappa(n),p}^{-1}(W,j,x)\,W(x)\\&\leqslant C(\kappa(n)/(\xi_{\kappa(n)}\,\mu_{\kappa(n)}))^{j+1/p}.\end{aligned} \tag{4.19}$$

Since the extreme right member of (4.19) is independent of x, (3.7) shows that (4.19) is valid for all $x\in\mathbb{R}$. Then (3.8) follows and (3.9) follows from the case $p=\infty$ and $j=1$ of (3.8). ▮

Remark. The case $j=0$ of Corollary 3.3 directly yields an L_∞ weighted-Nikolskii inequality: For all polynomials P of degree $\leqslant n$,

$$\|PW\|_{L_\infty(\mathbb{R})}\leqslant C(\kappa(n)/(\xi_n\mu_{\kappa(n)}))^{1/p}\,\|PW\|_{L_p(\mathbb{R})}.$$

Using this and a simple standard argument, one can show that for $0<r<p\leqslant\infty$, and all polynomials P of degree $\leqslant n$,

$$\|PW\|_{L_p(\mathbb{R})}\leqslant C(\kappa(n)/(\xi_n\mu_{\kappa(n)}))^{1/r-1/p}\,\|PW\|_{L_r(\mathbb{R})}.$$

Using different methods, Mhaskar and Saff [14, Theorems 3.1, 6.1] obtained weighted-Nikolskii inequalities in general situations.

5. Estimates Associated with Derivatives of Chebyshev Polynomials

In this section, we estimate, in effect, Freud–Christoffel functions for $[-1, 1]$. Throughout we let $T_k(x) = \cos(k \arccos(x))$ be the kth Chebyshev polynomial, $k = 0, 1, 2, \ldots$, and we let $v(x) = (1-x^2)^{-1/2}$ be the Chebyshev weight on $[-1, 1]$. The orthonormal polynomials for v are denoted by $p_k(v, x)$, $k = 0, 1, 2, \ldots$, so that $p_k(v, x) = (2/\Pi)^{1/2} T_k(x)$, $k = 1, 2, 3, \ldots$. We let

$$\left.\begin{aligned} K_n(v, x, t) &= \sum_{k=0}^{n-1} p_k(v, x)\, p_k(v, t) \\ &= (2/\Pi) \sum_{k=0}^{n-1}{}' \; T_k(x)\, T_k(t), \end{aligned}\right\} \tag{5.1}$$

where $'$ indicates that the first term is multiplied by $\frac{1}{2}$. Further, we let

$$\left.\begin{aligned} K_n'(v, x, t) &= \sum_{k=0}^{n-1} p_k'(v, x)\, p_k(v, t) \\ &= (2/\Pi) \sum_{k=0}^{n-1}{}' \; T_k'(v, x)\, T_k(v, t). \end{aligned}\right\} \tag{5.2}$$

Lemma 5.1. *Let $1 < p < \infty$. Then*

(i) $$\int_{-1}^{1} |K_n(v, x, t)|^p \, dt \leqslant C n^{p-1}, \qquad |x| \leqslant \tfrac{1}{2}. \tag{5.3}$$

(ii) $$\int_{-1}^{1} |K_n'(v, x, t)|^p \, dt \leqslant C n^{2p-1}, \qquad |x| \leqslant \tfrac{1}{2}. \tag{5.4}$$

Proof. Part (i) follows from Lemma 6.3.30 in Nevai [19, p. 120].

(ii) By the Christoffel–Darboux formula and (5.1) and (5.2), for $n \geqslant 2$,

$$K_n'(v, x, t) = (1/\Pi)(D_x\{T_n(x)/(x-t)\}\, T_{n-1}(t) - T_n(t)\, D_x\{T_{n-1}(x)/(x-t)\})$$

We deduce that

$$\begin{aligned} |K_n'(v, x, t)| &\leqslant (1/\Pi) \sum_{l=n-1}^{n} \left| \frac{d}{dx} \{T_l(x)/(x-t)\} \right| \\ &= (1/\Pi) \sum_{l=n-1}^{n} |T_l'(x)/(x-t) - T_l(x)/(x-t)^2| \\ &\leqslant C\, |x-t|^{-1} \max\{n, |x-t|^{-1}\}, \end{aligned}$$

if $|x| \leqslant \frac{1}{2}$, by the inequality (4.10). We write $[-1, 1] = \mathscr{I} \cup \mathscr{J}$, where $\mathscr{I} = \{t \in [-1, 1]\colon\ |x-t| \geqslant 1/n\}$ and $\mathscr{J} = [-1, 1] \backslash \mathscr{I}$. We see that as $|x-t|^{-1} \leqslant n$ in $\mathscr{I}$,

$$\int_{\mathscr{I}} |K_n'(v, x, t)|^p \, dt \leqslant Cn^p \int_{\mathscr{I}} |x-t|^{-p} \, dt$$

$$\leqslant Cn^p \int_{1/n}^{2} u^{-p} \, du \leqslant Cn^{2p-1}. \tag{5.5}$$

Further, by (4.10), for $|x| \leqslant \frac{1}{2}$, $|t| \leqslant 1$,

$$|K_n'(v, x, t)| \leqslant Cn \max_{|x| \leqslant 1} |K_n(v, x, t)| \leqslant Cn^2.$$

Hence

$$\int_{\mathscr{J}} |K_n'(v, x, t)|^p \, dt \leqslant Cn^{-1} n^{2p}. \tag{5.6}$$

Finally (5.5) and (5.6) yield (5.4). ∎

In accordance with standard usage, we shall say integers j and k have similar parity if they are either both even or both odd. Otherwise, they have opposite parity.

LEMMA 5.2. *Let j be a fixed positive integer. Then for all large enough k,*

$$\begin{aligned} T_k^{(j)}(0) &= 0, && \text{if } j, k \text{ have opposite parity} \\ &\sim (-1)^{(j+k)/2} k^j, && \text{if } j, k \text{ have similar parity.} \end{aligned} \tag{5.7}$$

Proof. Suppose first j is even. Then if k is odd, T_k is an odd polynomial and so $T_k^{(j)}(0) = 0$. Assume now k is even and write $k = 2l$. Using the explicit formula for the Chebyshev polynomials in Freud [4, p. 34, line 5], we see

$$T_k(x) = T_{2l}(x) = \sum_{n=0}^{l} B_{2l,n} \, x^{2n}, \tag{5.8}$$

where

$$B_{2l,n} = (-1)^{n+l} \sum_{\nu=0}^{n} \binom{2l}{2l-2\nu} \binom{l-\nu}{n-\nu}, \qquad n = 0, 1, 2, \ldots, l. \tag{5.9}$$

We omit the proof, as it is straightforward and tedious. Hence

$$(-1)^{n+l} B_{2l,n} \geqslant \binom{2l}{2l-2n}. \tag{5.10}$$

Then, by (5.8) and the formula for coefficients of a Maclaurin series, and taking $j = 2n$,

$$\begin{aligned}(-1)^{(j+k)/2} T_k^{(j)}(0) &= (-1)^{n+l} T_{2l}^{(2n)}(0) = (-1)^{n+l}(2n)!\, B_{2l,n} \\ &\geqslant \frac{(2l)!}{(2l-2n)!} \geqslant C(2l)^{2n} = Ck^j,\end{aligned}$$

by (5.10). Further by the inequality (4.10), we see

$$(-1)^{(j+k)/2}\, T_k^{(j)}(0) \leqslant Ck^j, \qquad k = j+1, j+2, \ldots.$$

This establishes (5.7) for j even. For j odd, the proof is similar. ∎

LEMMA 5.3. *Let j and l be fixed non-negative integers. Then for n large enough,*

$$\begin{aligned}\sum(n, j, l) &\triangleq \sum_{k=0}^{n-1}{}' \, T_k^{(j)}(0)\, T_k^{(l)}(0) \\ &= 0, \qquad \textit{if } j, l \textit{ have opposite parity.} \\ &\sim n^{j+l+1}, \qquad \textit{if } j, l \textit{ have similar parity.}\end{aligned} \tag{5.11}$$

Proof. By Lemma 5.2, $T_k^{(j)}(0)\, T_k^{(l)}(0) \neq 0$ only if j, k and l have the same parity. Hence $\sum(n, j, l) = 0$ if j, l have opposite parity. If j, k and l have the same parity, then $(j+k)+(l+k)$ is a multiple of 4, so that $(-1)^{(j+k)/2+(l+k)/2} = 1$. By Lemma 5.2,

$$\sum(n, j, l) \sim \sum k^{j+l},$$

where the second sum is over all non-negative $k \leqslant n$ such that k has the same parity as j and l. Then (5.11) follows. ∎

Finally, we construct a certain sequence of polynomials:

LEMMA 5.4. *Let j be a fixed non-negative integer. Let $0 < p \leqslant \infty$. Then there exists a positive integer r, and a sequence of polynomials Y_n, $n \geqslant n_0$, such that for $n \geqslant n_0$,*

(i) $$\deg(Y_n) \leqslant n-1.$$

(ii) $$\|Y_n\|_{L_p[-1,1]} \leqslant Cn^{r-1/p}. \tag{5.12}$$

(iii) *For* $s = 0, 1, 2, \ldots, j$,

$$\begin{aligned} Y_n^{(s)}(0) &= 0 && \textit{if } s \textit{ and } j \textit{ have opposite parity} \\ &\sim n^{r+s} && \textit{if } s \textit{ and } j \textit{ have similar parity}. \end{aligned} \tag{5.13}$$

Proof. Suppose first j is even. We use a trick from Nevai [19, p. 113, Theorem 6.3.13]. Let r be a positive integer such that $rp > 1$. For each positive integer n, let m denote the greatest integer $\leqslant (n-1)/r$. We set

$$Y_n(t) = \{K_m(v, 0, t)\}^r = \left\{\sum_{k=0}^{m-1}{}' \; T_k(0)\, T_k(t)/\Pi\right\}^r,$$

so that Y_n has degree $\leqslant n-1$. If $0 < p < \infty$, Lemma 5.1(i) yields

$$\int_{-1}^{1} |Y_n(t)|^p \, dt = \int_{-1}^{1} |K_m(v, 0, t)|^{rp} \, dt \leqslant Cn^{rp-1}.$$

Then (5.12) follows for $0 < p < \infty$. Since $|K_m(v, 0, t)| \leqslant Cn$, $|t| \leqslant 1$, (5.12) also follows for $p = \infty$. We next wish to prove (5.13). Let $0 \leqslant s \leqslant j$. By applying Leibniz's formula for the higher derivatives of a product of two functions repeatedly, we see

$$\begin{aligned} Y_n^{(s)}(t) &= D_t^s(\{K_m(v, 0, t)\}^r) \\ &= \sum \frac{s!}{l_1!\, l_2! \cdots l_r!} D_t^{l_1} K_m(v, 0, t)\, D_t^{l_2} K_m(v, 0, t) \cdots D_t^{l_r} K_m(v, 0, t), \end{aligned} \tag{5.14}$$

where the sum is over all r-tuples of non-negative integers $(l_1, l_2, \ldots, l_r)$ such that $l_1 + l_2 + \cdots + l_r = s$. By Lemma 5.3, the only non-zero terms in the sum when $t = 0$, are those for which $l_1, l_2, \ldots, l_r$ are all even. It follows that $Y_n^{(s)}(0) = 0$ if s is odd, since from $l_1 + l_2 + \cdots + l_r = s$, at least one l_i must be odd. Suppose now s is even. By Lemma 5.3, when $t = 0$, each non-zero term in this sum is bounded below by

$$Cm^{l_1+1} m^{l_2+1} \cdots m^{l_r+1} \geqslant Cn^{s+r}.$$

Hence $Y_n^{(s)}(0) \geqslant Cn^{s+r}$, where C depends on s, p and j. Finally inequality (4.10) shows that each term in the sum (5.14) is bounded above by Cn^{s+r}. This establishes (5.13).

If j is odd, one chooses an integer i such that $ip>1$, and sets $r=2i$. Further we let m be the greatest integer $\leqslant (n-1)/i$, and let

$$Y_n(t)=(K'_m(v,0,t))^i,$$

with the notation of (5.2). By Lemma 5.1(ii), if $0<p<\infty$,

$$\int_{-1}^{1} |Y_n(t)|^p\,dt \leqslant Cn^{2pi-1}=Cn^{pr-1}.$$

This yields (5.12). Using Lemma 5.3 as before, we see

$$Y_n^{(s)}(0)=0 \qquad \text{if } s \text{ is even and } s\leqslant j,$$

and

$$Y_n^{(s)}(0)\sim n^{s+2i}=n^{r+s} \qquad \text{if } s \text{ is odd and } s\leqslant j. \quad \blacksquare$$

One can use Lemma 5.4 to show that for $0<\varepsilon<1$ and $0<p<\infty$,

$$\inf_{P\in\mathbb{P}_{n-1}} \|P\|_{L_p[-1,1]}/|P^{(j)}(x)|\sim n^{-j-1/p},$$

$|x|\leqslant 1-\varepsilon$. However, we left Lemma 5.4 as it stands because we shall need the full detail of (5.13) in the next section.

6. Proof of the General Upper Bound

As a first step in establishing upper bounds for $\lambda_{n,p}(W,j,x)$, we need upper bounds for the $L_p(\mathbb{R})$ norm of a weighted polynomial PW in terms of the L_p norm of PW over some finite interval:

Lemma 6.1. *Let* $W(x)=\exp(-Q(x))$, *where* Q *satisfies* (2.1), (2.2), *and* (3.12). *Let* $0<p_1<\infty$. *Then if* q_n *is defined by* (3.13),

$$\|PW\|_{L_p(\mathbb{R})}\leqslant (1+CK^{-n})\,\|PW\|_{L_p(-Kq_n,Kq_n)}, \tag{6.1}$$

for all polynomials P *of degree* $\leqslant n$, *and for* $p_1\leqslant p\leqslant\infty$. *Here* $C=C(p_1)$ *only, while* $K\neq K(p_1,p)$.

Proof. As in [12], we use Cartan's lemma [2, p. 174], and modify a trick from the convergence theory of Padé approximation. Let $P\in\mathbb{P}_n$ and write

$$P(x)=c\prod_{j=1}^{m}(x-x_j)$$

where $m \leqslant n$, and where we may assume $c \neq 0$. We group the zeros of P as follows: For $1 \leqslant j \leqslant k$, $|x_j| \leqslant 2q_n$, and for $k < j \leqslant m$, $|x_j| > 2q_n$. Let $|x| \geqslant 2q_n$ and $|u| \leqslant q_n$. Then if $1 \leqslant j \leqslant k$,

$$|x - x_i|/|u - x_i| \leqslant 2\,|x|/|u - x_i|,$$

while if $k < j \leqslant m$,

$$|x - x_i|/|u - x_i| \leqslant (1 + |x|/(2q_n))/(1 - |u|/(2q_n)) \leqslant 2\,|x|/q_n.$$

Hence for $|x| \geqslant 2q_n$ and $|u| \leqslant q_n$,

$$\begin{aligned} |P(x)/P(u)| &\leqslant (2\,|x|/q_n)^{m-k}(2\,|x|)^k \Big/ \left|\prod_{j=1}^{k}(x - x_j)\right| \\ &\leqslant (2\,|x|/q_n)^{m-k}(2\,|x|/(rq_n))^k, \end{aligned} \tag{6.2}$$

for $|x| \geqslant 2q_n$ and $|u| \leqslant q_n$ such that $u \notin \mathscr{S}$, where $\mathscr{S}$ has linear Lebesgue measure at most $4erq_n$, and $r > 0$. Here we have used Cartan's lemma on small values of polynomials (see, for example, Baker [2, p. 174]). Let us choose $r = 1/(4e)$ and let $\mathscr{M} = (-q_n, q_n) \backslash (\mathscr{S} \cup (-A, A))$. Then $\mathscr{M}$ has linear Lebesgue measure at least $q_n - 2A$, for large n. Further if $|x| \geqslant 2q_n$, and $u \in \mathscr{M}$, (6.2) shows

$$|P(x)\,W(x)|/|P(u)\,W(u)| \leqslant \exp(n \log(8e\,|x|/q_n) - Q(x) + Q(u)). \tag{6.3}$$

We note that by (2.2), $M_1(\xi)$ is not identically zero, and hence $\xi M_1(\xi) \to \infty$ as $\xi \to \infty$. It follows that if, for example, $A \leqslant u \leqslant \xi$,

$$|Q(u)| \leqslant |Q(A)| + \int_A^\xi |Q'(u)|\,du \leqslant 2\xi M_1(\xi),$$

for large enough ξ. Hence, if n is large enough, (3.13) shows

$$|Q(u)| \leqslant 2q_n M_1(q_n) = 2n, \qquad u \in \mathscr{M}.$$

Together with (3.12) and (6.3), this implies that if $|x| \geqslant \max\{2, C_2\}\, q_n$ and $u \in \mathscr{M}$,

$$\begin{aligned} |P(x)\,W(x)|/|P(u)\,W(u)| &\leqslant \exp(n \log(8e^3) - 2n \log(|x|/q_n)) \\ &\leqslant \exp(-n \log(|x|/q_n)) \end{aligned}$$

if $|x| \geqslant Kq_n$, with K large enough. Let $p_1 \leqslant p < \infty$. We have

$$\int_{|x| \geqslant Kq_n} |P(x)\, W(x)|^p \, dx$$

$$\leqslant \int_{|x| \geqslant Kq_n} (q_n/|x|)^{np} dx \inf\{|P(u)\, W(u)|^p : u \in \mathcal{M}\}$$

$$\leqslant 2q_n(np-1)^{-1}K^{1-np}(q_n-2A)^{-1} \int_{\mathcal{M}} |P(u)\, W(u)|^p \, du.$$

(as $\mathbb{M}$ has linear measure at least $q_n - 2A$)

$$\leqslant K^{-np} \int_{-q_n}^{q_n} |P(u)\, W(u)|^p \, du$$

if $p_1 \leqslant p < \infty$, and $n \geqslant n_0$, where $n_0 = n_0(p_1)$ only. Finally (6.1) follows easily from this last inequality for $p_1 \leqslant p < \infty$. Because the constants are independent of p, we can let $p \to \infty$ to deduce (6.1) for $p = \infty$ also. ▮

In much the same way as in Lemma 4.2, one can prove that q_x is non-decreasing; $\lim_{x \to \infty} q_x = \infty$, x/q_x is non-decreasing and $q_{2x}/q_x \leqslant 2$. These basic properties of q_x are implicit in the sequel.

LEMMA 6.2. *Let $F(y)$ be a function of y, and y be a function of x. Let k be a positive integer. Then*

$$D_x^k\{F(y)\} = \sum \frac{k!}{i_1!\, i_2! \cdots i_q!} F^{(l)}(y) \prod_{s=1}^{q} (y^{(s)}(x)/s!)^{i_s}, \tag{6.4}$$

where the sum is taken over all integers $q > 0$, $i_s \geqslant 0$, $s = 1, 2, \ldots, q$, such that $i_1 + 2i_2 + 3i_3 + \cdots + qi_q = k$. Further, $l = i_1 + i_2 + \cdots + i_q$.

Proof. See Gradshteyn and Ryzhik [10, p. 19, formula 0.430(2)]. ▮

We can now construct polynomials which approximate $1/W$:

LEMMA 6.3. *Let $W(x) = \exp(-Q(x))$, where Q satisfies* (2.1), (2.2) *with* $A = 0$, (3.10) *and* (3.12). *Let K be the constant in Lemma* 6.1. *Let*

$$B_n = \min\{0, \min\{Q''(u) : |u| \leqslant Kq_n\}\}, \tag{6.5}$$

$$U_n[t; x] = Q'(x)(t-x) + B_n(t-x)^2/2, \tag{6.6}$$

and, with the notation of (4.1), *let*

$$V_n[t; x] = W^{-1}(x)\, P_m(U_n[t; x]), \tag{6.7}$$

where m is the greatest integer $\leqslant n/4$. Let j be a positive integer. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that for $n \geqslant n_0$, $|x| \leqslant \delta q_n$ and $|t| \leqslant Kq_n$, the following hold:

(i) $V_n[t; x]$ *has degree* $\leqslant n/2$.

(ii) $$0 < V_n[t; x]\, W(t) \leqslant 5/4. \tag{6.8}$$

(iii) $$V_n[x; x] = W^{-1}(x). \tag{6.9}$$

(iv) $$|D_x^k V_n[t; x]| \leqslant \varepsilon W^{-1}(x)(n/q_n)^k, \qquad k = 1, 2, ..., j. \tag{6.10}$$

Proof. Part (i) follows from (6.7) and the choice of m.

(ii) Let $|x| \leqslant Kq_n$, $|t| \leqslant Kq_n$. By (6.6), for some w between t and x,

$$Q(t) - Q(x) - U_n[t; x] = (Q''(w) - B_n)(t - x)^2/2 \geqslant 0,$$

by choice of B_n. Hence

$$W^{-1}(t) \geqslant W^{-1}(x) \exp(U_n[t; x]). \tag{6.11}$$

Let $0 < \delta < 1$. If $|x| \leqslant \delta q_n$ and $|t| \leqslant Kq_n$, (6.6) shows

$$\begin{aligned} |U_n[t; x]| &\leqslant 2KM_1(\delta q_n)\, q_n + 2\, |B_n|\, (Kq_n)^2 \\ &\leqslant 2KM_1(\delta q_n)\, q_n + 2\delta(M_1(q_n)/q_n)(Kq_n^2) \end{aligned}$$

(by (3.10) and (6.5), if n is large enough)

$$\leqslant M_1(q_n)\, q_n/17 = n/17,$$

if δ is small enough, by (3.11) and (3.13). Hence from Lemma 4.1 and (6.11), we deduce that for $|t| \leqslant Kq_n$ and $|x| \leqslant \delta q_n$, and n large enough

$$W^{-1}(t) \geqslant (4/5)\, W^{-1}(x)\, P_m(U_n[t; x]) > 0$$

and (6.8) follows.

Part (iii) follows immediately from (6.6) and (6.7).

(iv) For large enough n,

$$P_m(t) = \exp(t) + O(t^{j+1}) \qquad \text{as } t \to 0.$$

Further $U_n[t; x] = O(|t - x|)$ as $t \to x$. Hence, as $t \to x$

$$P_m(U_n[t; x]) = \exp(U_n[t; x]) + O(|t - x|^{j+1})$$

and from (6.7), we deduce

$$D_x^k V_n[t; x] = W^{-1}(x)\, D_x^k \exp(U_n[t; x]), \qquad k = 1, 2, ..., j. \tag{6.12}$$

We now apply Lemma 6.2 to estimate the right member of (6.12). Let $F(y)=\exp(y)$ and $y=U_n[t;x]$, so that as $y(x)=0$, $F^{(l)}(y)=1$ in (6.4). Further $y^{(s)}(x)=0$ if $s>2$. It follows that

$$D_x^k\{\exp(U_n[t;x])\}=\sum\frac{k!}{i_1!\,i_2!}\prod_{s=1}^{2}(D_x^s U_n[t;x]/s!)^{i_s}, \tag{6.13}$$

where the sum is over all $i_1\geqslant 0$, $i_2\geqslant 0$ such that $i_1+2i_2=k$. Now if $|x|\leqslant\delta q_n$, where δ is small enough, (3.11) and (6.6) show that

$$|D_x U_n[t;x]|\leqslant M_1(\delta q_n)\leqslant\varepsilon M_1(q_n)=\varepsilon n/q_n,$$

while if $n\geqslant n_0(\varepsilon)$, (3.10), (6.5) and (6.6) show that

$$|D_x^2 U_n[t;x]|=|B_n|\leqslant\varepsilon M_1(q_n)/q_n=\varepsilon n/q_n^2.$$

Hence for $|x|\leqslant\delta q_n$ and $|t|\leqslant Kq_n$, (6.13) yields

$$|D_x^k\{\exp(U_n[t;x])\}|\leqslant C\varepsilon(n/q_n)^k$$

and combined with (6.12), this inequality yields (6.10). ∎

Proof of Theorem 3.4. In much the same way as in Theorem 3.1, we may assume $A=0$ in (2.2). Given a positive integer n, we let l denote the greatest integer $\leqslant n/2$. By (2.4) and Lemma 6.1, for $n\geqslant n_0$,

$$\lambda_{n,p}(W,j,x)\leqslant 2\inf_{P\in\mathbb{P}_{n-1}}\|PW\|_{L_p(-Kq_n,Kq_n)}/|P^{(j)}(x)|$$

$$\leqslant 2\inf_{P^*\in\mathbb{P}_{l-1}}\|P^*(t)V_n[t;x]\,W(t)\|_{L_p(-Kq_n,Kq_n)}/|D_x^j\{P^*(t)V_n[t;x]\}|$$

(where $V_n[t;x]$ is as in Lemma 6.3, and by Lemma 6.3(i))

$$\leqslant(10/8)\inf_{\mathbb{P}_{l-1}}\|P^*(t)\|_{L_p(-Kq_n,Kq_n)}/|D_x^j\{P^*(t)\,V_n[t;x]\}|, \tag{6.14}$$

by (6.8). Now choose

$$P^*(t)=Y_l((t-x)/(2Kq_n)), \tag{6.15}$$

where Y_l is as in Lemma 5.4. If $|x|\leqslant Kq_n$, we see that

$$\|P^*(t)\|_{L_p(-Kq_n,Kq_n)}\leqslant(2Kq_n)^{1/p}\,\|Y_l\|_{L_p[-1,1]}$$
$$\leqslant Cq_n^{1/2}n^{r-1/p}, \tag{6.16}$$

by (5.12). Let $|x| \leqslant \delta q_n$, where δ is so small that (6.10) holds for a given ε. Then Leibniz's formula shows that

$$|D_x^j\{P^*(t)\, V_n[t;x]\}|$$

$$\geqslant |P^{*(j)}(x)\, V_n[x;x]| - \sum_{k=1}^{j} \binom{j}{k} |P^{*(j-k)}(x)|\, |D_x^k V_n[t;x]|$$

$$\geqslant W^{-1}(x)(C_1 q_n^{-j}\, |Y_l^{(j)}(0)| - C_2 \varepsilon \sum_{k=1}^{j} |Y_l^{(j-k)}(0)|\, q_n^{-(j-k)}(n/q_n)^k)$$

(by (6.9), (6.10) and (6.15))

$$\geqslant W^{-1}(x)\, n^r (n/q_n)^j (C_3 - C_4 \varepsilon), \tag{6.17}$$

by (5.13), where C_3, C_4 depend on j, but are independent of ε. If ε is small enough, $C_3 - C_4\varepsilon > 0$ and so for $|x| \leqslant \delta q_n$, (6.14), (6.16), and (6.17) yield (3.14). ∎

We remark that if Q is even and convex, and $Q'(x) \geqslant 0$, $x \in (0, \infty)$, then the proofs above can be substantially simplified: In (6.5), $B_n = 0$, and all non-vanishing derivatives of $V_n[t;x]$ have the same sign. Further then, there is no longer any necessity to impose (3.11).

7. Proof of Theorems 3.5 and 3.6

Theorem 3.5 will be deduced from Theorems 3.1 and 3.4, but first we need some preliminary lemmas and notation. Throughout, we let $f_0(x) = 1$, and given a positive integer j, we let

$$f_j(x) = \prod_{k=1}^{j} \exp_k(x), \qquad x \in \mathbb{R}, \tag{7.1}$$

$$g_j(x) = \prod_{k=1}^{j} \log_k(x), \qquad x > \exp_{j-1}(0). \tag{7.2}$$

By induction on j it is not difficult to see that

$$D_x \exp_j(x) = f_j(x), \qquad x \in \mathbb{R}, \tag{7.3}$$

and

$$D_x f_j(x) = f_j(x) \sum_{k=1}^{j} f_{k-1}(x), \qquad x \in \mathbb{R}. \tag{7.4}$$

LEMMA 7.1. *Let $c, \rho > 0$ and l be a fixed positive integer. Let*

$$Q(x) = c \exp_l(|x|^\rho), \qquad x \in \mathbb{R}, \tag{7.5}$$

and let

$$h(x) = \rho x^{\rho-1} f_{l-1}(x^\rho), \qquad x > 0. \tag{7.6}$$

Then

(i) $Q^{(j)}(x) = Q(x)(h(x))^j (1 + o(1)), \qquad x \to \infty, j = 1, 2, 3.$ (7.7)

(ii) *If ξ_x is the root of* (3.1), *then*

$$\xi_x^\rho = \log_l(x\eta_x/\{c\rho^2 g_l^2(x)\}), \qquad x \to \infty, \tag{7.8}$$

where

$$\eta_x = 1 + o(1), \qquad x \to \infty. \tag{7.9}$$

(iii) $1 - \xi_x/\xi_{2x} \sim (g_l(x))^{-1}, \qquad x \to \infty.$ (7.10)

Proof. (i) For $j = 1$, we see from (7.1), (7.3), (7.5), and (7.6), that

$$Q'(x) = Q(x)\, h(x). \tag{7.11}$$

Differentiating, we obtain, using (7.11),

$$\begin{aligned} Q''(x) &= Q'(x)\, h(x) + Q(x)\, h'(x) \\ &= Q(x)\, h^2(x)\{1 + h'(x)/h^2(x)\}, \end{aligned} \tag{7.12}$$

and similarly

$$Q'''(x) = Q(x)\, h^3(x)\{1 + 3h'(x)/h^2(x) + h''(x)/h^3(x)\}. \tag{7.13}$$

If we compute $h'(x)$ and $h''(x)$ from (7.6) and (7.4) (we omit the messy formulae) and use the fact that (if $l \geqslant 2$) $\lim_{x\to\infty} x^a f_{l-2}(x^\rho)/\exp_{l-1}(x^\rho) = 0$, for any $a > 0$, then we see that

$$\lim_{x\to\infty} h'(x)/h^2(x) = \lim_{x\to\infty} h''(x)/h^3(x) = 0, \tag{7.14}$$

and then (7.12) and (7.13) imply (7.7).

(ii) For large positive x, we may obviously define $\eta_x > 0$ by the formula in (7.8). We must prove (7.9). Now by (3.1) and by (7.7) with $j=2$, as $x \to \infty$,

$$x = \xi_x^2 Q''(\xi_x) = \xi_x^2 Q(\xi_x)(h(\xi_x))^2(1+o(1))$$
$$= c\exp_l(\xi_x^\rho)(\rho\xi_x^\rho f_{l-1}(\xi_x^\rho))^2 (1+o(1))$$

(by (7.5) and (7.6))

$$= cx\eta_x\{c\rho^2 g_l^2(x)\}^{-1}\left\{\rho \prod_{k=1}^{l} \log_k(x\eta_x/\{c\rho^2 g_l^2(x)\})\right\}^2 (1+o(1)),$$

by (7.1) and (7.8). Using (7.2), we deduce

$$1 = \eta_x \prod_{k=1}^{l} \{\log_k(x\eta_x/\{c\rho^2 g_l^2(x)\})/\log_k x\}^2 (1+o(1)). \tag{7.15}$$

If $\eta_x < 1-\delta$, the right member of (7.15) is bounded above by $\eta_x(1+o(1))$ for large x, and so $1 \leqslant \eta_x < 1$, a contradiction. Thus $\eta_x \geqslant 1-o(1)$ for large x. Then for large x, the right member of (7.15) is bounded below by $\eta_x(1+o(1))$ and so $1 \geqslant \eta_x(1+o(1))$ and $\eta_x \geqslant 1$. Hence (7.9) follows.

(iii) Firstly, differentiating (3.1) with respect to x,

$$2\xi_x'\xi_x Q''(\xi_x) + \xi_x^2 Q'''(\xi_x)\,\xi_x' = 1$$
$$\Rightarrow x\xi_x'/\xi_x\{2 + Q'''(\xi_x)\,\xi_x/Q''(\xi_x)\} = 1, \tag{7.16}$$

by (3.1). Now by (7.7) and (7.6), as $x \to \infty$,

$$Q'''(\xi_x)\,\xi_x/Q''(\xi_x) = \rho\xi_x^\rho f_{l-1}(\xi_x^\rho)(1+o(1))$$
$$= \rho(\log_l x) \prod_{k=1}^{l-1} \log_{l-k}(x)(1+o(1))$$

(by (7.1), (7.8), and (7.9))

$$= \rho g_l(x)(1+o(1)).$$

From (7.16), we deduce

$$x\xi_x'/\xi_x = (\rho g_l(x))^{-1}(1+o(1)), \qquad x \to \infty. \tag{7.17}$$

Next, from (7.2), (7.8), and (7.9), we see that for all large enough x, and $x \leqslant v \leqslant 2x$,

$$\xi_x \sim \xi_v \sim \xi_{2x} \qquad \text{and} \qquad g_l(x) \sim g_l(v). \tag{7.18}$$

Then, given large positive x, there exists v between x and $2x$ such that

$$\begin{aligned} 1-\xi_x/\xi_{2x} &= (\xi_{2x}-\xi_x)/\xi_{2x} \\ &= x\xi'_v/\xi_{2x} \\ &\sim v\xi'_v/\xi_v \\ &\text{(by (7.18))} \\ &\sim (g_l(v))^{-1} \\ &\text{(by (7.17))} \\ &\sim (g_l(x))^{-1}, \end{aligned}$$

by (7.18). ▮

LEMMA 7.2. *Let $Q(x)$ be given by (7.5). For each positive integer n, let $a_n = a$ be the root of the equation*

$$n = (2/\Pi)\int_0^1 axQ'(ax)(1-x^2)^{-1/2}dx. \tag{7.19}$$

Then

(i) *For all polynomials P of degree $\leqslant n$,*

$$\|PW\|_{L_\infty(\mathbb{R})} = \|PW\|_{L_\infty(-a_n, a_n)}. \tag{7.20}$$

(ii) *There exists a constant C such that for large n,*

$$a_n^\rho \leqslant \log_l(Cn\{g_l(n)\}^{-1/2}). \tag{7.21}$$

Proof. Part (i) follows from (3.7) in Example 3 in Mhaskar and Saff [15].

(ii) We apply a rather crude form of Laplace's method to (7.19). A more detailed analysis shows that the right member of (7.21) is of the correct order as regards the power of $g_l(n)$. Firstly, we can rewrite (7.19) as

$$(\Pi n/(2a)) = \int_0^{\Pi/2} \cos\theta Q'(a\cos\theta)\, d\theta. \tag{7.22}$$

Given $a > 0$, let

$$\varepsilon = (ah(a)/2)^{-1/2}, \tag{7.23}$$

where h is as in (7.6). Further, let

$$f(\theta) = f(a;\theta) = \log Q'(a\cos\theta), \qquad \theta\in[0, \Pi/2]. \tag{7.24}$$

Then (7.22) yields

$$(\Pi n/2a)) \geqslant \cos \varepsilon \int_0^{\varepsilon} \exp(f(\theta))\, d\theta. \tag{7.25}$$

For notational convenience, let

$$v = v(a; \theta) = a \cos \theta \qquad \text{and} \qquad w = w(a; \theta) = a \sin \theta. \tag{7.26}$$

Differentiating (7.24), we obtain

$$f'(\theta) = -wQ''(v)/Q'(v),$$
$$f''(\theta) = w^2\{Q'''(v)\,Q'(v) - (Q''(v))^2\}/(Q'(v))^2 - vQ''(v)/Q'(v).$$

By (7.7), we have for $0 \leqslant \theta \leqslant \varepsilon$,

$$f''(\theta) = w^2\{h^2(v)(1+o(1) - h^2(v)(1+o(1))\} - vh(v)(1+o(1))$$
$$= O(a^2\varepsilon^2)\{o(h^2(a))\} - vh(v)(1+o(1))$$

(by (7.26) and monotonicity of h)

$$= o(\varepsilon^{-2}) - vh(v)(1+o(1)), \tag{7.27}$$

by (7.23). Next, given $0 \leqslant \theta \leqslant \varepsilon$, there exists η between 0 and θ such that

$$vh(v) = a \cos \theta h(a \cos \theta)$$
$$= ah(a) + \theta(-a(\sin \eta)\, h(a \cos \eta) - a^2(\cos \eta)(\sin \eta)\, h'(a \cos \eta))$$
$$= ah(a) + O(a\varepsilon^2 h(a)) + o(a^2\varepsilon^2 h^2(a))$$

(as $0 \leqslant \eta \leqslant \varepsilon$ and by (7.14))

$$= ah(a) + o(\varepsilon^{-2}).$$

Hence, for $0 \leqslant \theta \leqslant \varepsilon$, (7.27) and this last inequality show

$$f''(\theta) = -ah(a) + o(\varepsilon^{-2}). \tag{7.28}$$

Next for $0 \leqslant \theta \leqslant \varepsilon$, there exists η between 0 and θ such that

$$f(\theta) = f(0) + \theta f'(0) + (\theta^2/2) f''(\eta)$$
$$= \log Q'(a) + 0 - (ah(a)/2)\, \theta^2 + o(\theta^2\varepsilon^{-2})$$

(by (7.24) and (7.28))

$$= \log Q'(a) - (\theta/\varepsilon)^2 + o(1),$$

by (7.23). Hence for large n and $a = a_n$, (7.25) yields

$$(\prod n/a) \geqslant Q'(a) \int_0^{\varepsilon} \exp(-(\theta/\varepsilon)^2)\, d\theta$$
$$= Q'(a)\varepsilon \int_0^1 \exp(-u^2)\, du$$

Hence, for some constant C_1 independent of n,

$$C_1 n \geqslant aQ'(a)\,\varepsilon$$
$$= Q(a)(ah(a))(ah(a)/2)^{-1/2}$$

(by (7.11) and (7.23))

$$\geqslant c \exp_l(a^\rho)\{\rho a^\rho f_{l-1}(a^\rho)\}^{1/2},$$

by (7.5) and (7.6). Writing

$$a_n^\rho = a^\rho = \log_l(\delta_n n/\{g_l(n)\}^{1/2}),$$

where $\delta_n > 0$, we see that $\delta_n \leqslant C$, $n \geqslant n_0$, in much the same way as in the proof of Lemma 7.1(ii). This establishes (7.21). ▌

We can now prove Theorem 3.5:

Proof of Theorem 3.5. Firstly, from (3.16) and (7.8), we see

$$\xi_n = \theta_n(1 + o(1)), \qquad n \to \infty. \tag{7.29}$$

Next, taking $s = 1$ in Theorem 3.1, (3.3) and (7.10) show

$$\mu_n \sim (g_l(n))^{-1/2}, \qquad n \to \infty. \tag{7.30}$$

Hence Theorem 3.1 shows that for $0 < \varepsilon < 1$,

$$\lambda_{n,p}(W, j, x) \geqslant C(\theta_n/n)^{j+1/p}\, W(x), \qquad |x| \leqslant \varepsilon\theta_n, \tag{7.31}$$

while

$$\lambda_{n,p}(W, j, x) \geqslant C(\theta_n\{g_l(n)\}^{-1/2}/n)^{j+1/p}\, W(x), \qquad |x| \leqslant \xi_n. \tag{7.32}$$

Next, let us set

$$\kappa(n) = C_1 n(g_l(n))^{3/2}, \tag{7.33}$$

for large enough n, where C_1 is some positive constant. By (7.8) and (7.9) we see

$$\xi_{\kappa(n)}^{\rho} = \log_l(C_1 n(g_l(n))^{3/2}(1+o(1))/\{c\rho^2 g_l^2(\kappa(n))\})$$
$$= \log(C_1 n(g_l(n))^{-1/2}(1+o(1))/\{c\rho^2\})$$

(by (7.2) and (7.33))

$$\geqslant a_n^{\rho},$$

if C_1 is large enough, by (7.21). Then Lemma 7.2(i) shows that (3.7) holds. Hence (3.8) and (3.9) hold. But, by (7.29), (7.30), and (7.33)

$$\xi_n \mu_{\kappa(n)}/\kappa(n) \sim \theta_n g_l(\kappa(n))^{-1/2} n^{-1} g_l(n)^{-3/2}$$
$$\sim \theta_n/(n v_n),$$

by (3.18), (7.2), and (7.33). Together with (3.8) and (3.9), (7.31) and (7.32) yield the lower bound in (3.17) and also yield (3.19), (3.20), and (3.21). Finally, we must establish the upper bound in (3.17), and to this end we apply Theorem 3.4. Since Q is convex, (3.10) holds trivially, while it is easy to verify that (3.11) holds. Further, if $|x| \geqslant 2^{1/\rho}\xi, Q(x) \geqslant Q(\xi)^2$, and then (3.12), follows easily from (7.5), (7.6), and (7.7). Finally, from (3.13), (7.5), (7.6), and (7.7), we obtain $q_n = \theta_n(1+o(1))$, $n \to \infty$. Then (3.14) yields (3.17). ∎

Proof of Theorem 3.6(i). From (3.1), (3.3), and (3.13), we see

$$\xi_x = (x\{\alpha/(\alpha-1)\})^{1/\alpha},$$
$$\mu_x = (1-2^{-1/\alpha})^{1/2},$$
$$q_x = (x/\alpha)^{1/\alpha},$$

and it is easy to see that (3.10), (3.11), and (3.12) are satisfied. Then Theorems 3.1 and 3.4 immediately yield (3.23). Further, by Lemma 6.1, there exists $C > 0$ such that

$$\|PW\|_{L_\infty(\mathbb{R})} \leqslant 2\,\|PW\|_{L_\infty(-Cn^{1\alpha}, Cn^{1\alpha})},$$

for all polynomials P of degree $\leqslant n$. If follows that in (3.7), we may take $\kappa(n) = C_1 n$, where C_1 is large enough. Then (3.8) yields (3.24). ∎

Proof of Theorem 3.6(ii). The upper bound (3.25) follows as before from Theorem 3.4. The lower bounds may be proved as follows: Let

$$Q^*(u) = |u|^{\alpha/2}, \qquad u \in \mathbb{R}.$$

Note that as $\alpha < 2$, $Q^{*''}(u) < 0$, $u > 0$. Given $x \neq 0$, let

$$S[u; x] = Q^{*'}(x^2)(u - x^2) = (\alpha/2)\,|x|^{\alpha-2}(u - x^2).$$

Then for $0 < u < \infty$, there exists v between u and x^2 such that

$$Q^*(u) - Q^*(x^2) - S[u; x] = Q^{*''}(v)(u - x^2)^2/2! \leqslant 0,$$

and we deduce, by setting $u = t^2$, that

$$Q(t) \leqslant Q(x) + S[t^2; x], \qquad t \in \mathbb{R}. \tag{7.34}$$

Now let $K > \varepsilon > 0$ and let $\varepsilon n^{1/\alpha} \leqslant |x| \leqslant K n^{1/\alpha}$. Then for $|t| \leqslant 2Kn^{1/\alpha}$,

$$\begin{aligned}|S[t^2; x]| = (\alpha/2)\,|x|^\alpha\,|(t/x)^2 - 1| &\leqslant (\alpha/2)\,K^\alpha n 8(K/\varepsilon)^2\\ &\leqslant C_1 n/4.\end{aligned}$$

Here $C_1 = C_1(\alpha, \varepsilon, K)$. Then if m is the least integer $\geqslant C_1 n/4$, (7.34) and (4.2) show

$$\begin{aligned}W(t) &\geqslant W(x)\exp(-S[t^2; x])\\ &\geqslant (4/5)\,W(x)\,P_m(-S[t^2; x]) > 0,\end{aligned} \tag{7.35}$$

for $\varepsilon n^{1/\alpha} \leqslant |x| \leqslant K n^{1/\alpha}$ and $|t| \leqslant 2Kn^{1/\alpha}$. Hence if

$$R_n[t; x] = W(x)\,P_m(-S[t^2; x]),$$

we obtain from (7.35),

$$\begin{aligned}&\lambda_{n,p}(W, 0, x)\\ &\quad\geqslant (4/5)\,W(x)\inf_{\mathbb{P}_{n-1}}\|P(t)\,R_n[t; x]\|_{L_p(-2Kn^{1/\alpha}, 2Kn^{1/\alpha})}/|P(x)\,R_n[x, x]|\\ &\quad\geqslant C W(x)(n^{1/\alpha}/n)^{1/p},\end{aligned}$$

by Lemma 4.5, if $\varepsilon n^{1/\alpha} \leqslant |x| \leqslant K n^{1/\alpha}$.

Further, from the proof of Lemma 6.1 or the proof of Lemma 6.3 in [14], it is not difficult to see that for large enough C_2 and $|x| \geqslant C_2 q_n = C_2(n/\alpha)^{1/\alpha}$,

$$\|PW\|_{L_p(\mathbb{R})}/|PW|(x) \geqslant C.$$

Hence (3.26) holds for $j = 0$ and $|x| \geqslant \varepsilon n^{1/\alpha} = \varepsilon\theta_n$. The proof for general j may be completed by induction in much the same way as the proof of

Theorem 3.1. The only difference is that, instead of an inequality like (4.18), one uses the following: If $\varepsilon n^{1/\alpha} \leqslant x \leqslant K n^{1/\alpha}$,

$$|P'(x)\, W(x)| = |P'(x)\, R_n[x; x]|$$
$$\leqslant C(n/\xi_n) \max\{|PW|(x): (\varepsilon/2)\, n^{1/\alpha} \leqslant x \leqslant 2Kn^{1/\alpha}\}.$$

Similarly if $-Kn^{1/\alpha} \leqslant x \leqslant -\varepsilon n^{1/\alpha}$. ∎

The proof of Theorem 3.6(iii) requires a few lemmas.

LEMMA 7.3. *Let* $W(x) = \exp(-Q(x))$, $x \in \mathbb{R}$, *where* $Q(x) = |x|^\alpha$, $x \in \mathbb{R}$, *some* $0 < \alpha < 1$. *Let* $\beta > 0$, $0 < p \leqslant \infty$ *and* j *be a non-negative integer. Then*

$$W^{-\beta}(x)\, \lambda_{n,p}(W^\beta, j, x) \geqslant \lambda_{n,p}(W^\beta, j, 0), \qquad x \in \mathbb{R}. \tag{7.36}$$

Proof. We use the argument of Mhaskar and Saff [14, Theorem 6.5(a)]: Since $Q(t-x) \leqslant Q(t) + Q(x)$, $t, x \in \mathbb{R}$, we have

$$W(t-x)/W(x) \geqslant W(t), \qquad x, t \in \mathbb{R}. \tag{7.37}$$

Let $0 < p < \infty$. By (2.4), if the inf's are over $\mathbb{P}_{n-1}$,

$$\{W^{-\beta}(x)\, \lambda_{n,p}(W^\beta, j, x)\}^p$$
$$= \inf \int_{-\infty}^{\infty} |P(u)\, W^\beta(u)|^p\, du / |P^{(j)}(x)\, W^\beta(x)|^p$$
$$= \inf \int_{-\infty}^{\infty} |P(t-x)\, W^\beta(t-x)|^p\, dt / |P^{(j)}(x)\, W^\beta(x)|^p$$
$$\geqslant \inf \int_{-\infty}^{\infty} |R(t)\, W^\beta(t)|^p\, dt / |R^{(j)}(0)|^p,$$

where $R(t) = P(t-x)$ and we have used (7.37). Then (7.36) follows. For $p = \infty$, the proof is easier. ∎

LEMMA 7.4. *With the notation of Lemma* 7.3,

$$W^{2\beta}(x) \sum_{k=0}^{\infty} (p_k^{(j)}(W^{2\beta}, x))^2 \leqslant C, \qquad x \in \mathbb{R}, \tag{7.38}$$

where $C = C(j, \beta)$.

Proof. In the proof of this lemma, we let $p_k(x) = p_k(W^{2\beta}; x)$. By (2.5) and (7.36) it suffices to prove

$$\sum_{k=0}^{\infty} (p_k^{(j)}(0))^2 < \infty. \tag{7.39}$$

First, we note that

$$\int_{-\infty}^{\infty} (\log W^{2\beta}(u))/(1+u^2)\,du = -2\beta \int_{-\infty}^{\infty} |u|^{\alpha}/(1+u^2)\,du > -\infty,$$

and so the moment problem associated with $W^{2\beta}$ is indeterminate (Akhiezer [1, pp. 87–88, Problem 14]). Then Akhiezer [1, pp. 5, 54] shows that

$$\sum_{k=0}^{\infty} |p_k(z)|^2$$

converges uniformly in compact subsets of $\mathbb{C}$. By the Cauchy–Schwarz inequality, it follows that

$$\sum_{k=0}^{\infty} |p_k(z)\,p_k(u)|$$

converges uniformly for z, u in compact subsets of $\mathbb{C}$. Applying Cauchy's integral formula for derivatives of analytic functions twice, we obtain

$$\sum_{k=0}^{n} (p_k^{(j)}(0))^2 = (j!/(2\Pi i))^2 \int_{|z|=1} z^{-j-1} \times \int_{|u|=1} \left(\sum_{k=0}^{n} p_k(z)\,p_k(u)\right) u^{-j-1}\,du\,dz$$

and we deduce (7.39). ∎

Proof of Theorem 3.6(iii). From Lemma 7.4 with $j=0$ and a suitable choice of β, and from Theorem 6.1 in Mhaskar and Saff [14], we deduce that if $0<p<r\leqslant\infty$, then for all $P\in\mathbb{P}_n$,

$$\|PW\|_{L_r(\mathbb{R})} \leqslant C\,\|PW\|_{L_p(\mathbb{R})}, \tag{7.40}$$

where $C=C(p,r)$. In particular, taking $r=\infty$, and using (2.4), we see

$$W^{-1}(x)\,\lambda_{n,p}(W,0,x) \geqslant C_1, \qquad x\in\mathbb{R},$$

which establishes (3.27) for $j=0$ and $0<p\leqslant\infty$. Next, let $0<p\leqslant 2$ and j be a non-negative integer. Let P be a polynomial of degree $\leqslant n$. Let

$$a_k = \int_{-\infty}^{\infty} P(t)\,p_k(W^2;t)\,W^2(t)\,dt, \qquad k=0,1,2,\ldots,n.$$

Then

$$|P^{(j)}W|(x)=\left|\sum_{k=0}^{n} a_k p_k^{(j)}(W^2;x)\,W(x)\right|$$
$$\leqslant C\left(\sum_{k=0}^{n} a_k^2\right)^{1/2}$$

(by the Cauchy–Schwarz inequality and Lemma 7.4)

$$=C\,\|PW\|_{L_2(\mathbb{R})}$$
$$\leqslant C_1\,\|PW\|_{L_p(\mathbb{R})},$$

by (7.40). We deduce that (3.27) holds for $0<p\leqslant 2$, and $j=0, 1, 2,\ldots$ ∎

Finally, we prove that the polynomials are not dense in $\Lambda_p=\{f: fW\in L_p(\mathbb{R})\}$, $0<p<\infty$, and W as in Theorem 3.6(iii). We note that for more general weights, a related result is quoted in Akhiezer [1, p. 87, problem 13].

LEMMA 7.5. *Let* $W(x)=\exp(-Q(x))$, $x\in\mathbb{R}$, *where* $Q(x)=|x|^\alpha$, *some* $0<\alpha<1$. *Then if* $0<p<\infty$, *the polynomials are not dense in* Λ_p.

Proof. Suppose the contrary. Then for each $f\in\Lambda_p$, there exists a sequence of polynomials $\{U_n\}$ such that

$$\lim_{n\to\infty}\|(f-U_n)\,W\|_{L_p(\mathbb{R})}=0. \tag{7.41}$$

Consider some finite subinterval of $\mathbb{R}$, say $[0, 1]$. Then (7.41) implies that $U_n W$ converges in linear Lebesgue measure in $[0, 1]$ to fW as $n\to\infty$, and consequently some subsequence converges a.e. in $[0, 1]$ to fW. We shall denote this subsequence by $\{U_n W\}$ also. Now by (7.40), there exists C independent of n and of $\{U_n\}$ such that

$$\|U_n W\|_{L_\infty(\mathbb{R})}\leqslant C\,\|U_n W\|_{L_p(\mathbb{R})}.$$

We deduce that for a.e. $x\in[0, 1]$,

$$|fW|(x)\leqslant C\,\|fW\|_{L_p(\mathbb{R})}$$

and hence for all $f\in\Lambda_p$, and some C independent of f,

$$\|fW\|_{L_\infty[0,1]}\leqslant C\,\|fW\|_{L_p(\mathbb{R})},$$

which is obviously false. ∎

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